The stability of vacua in two-dimensional gauge theory

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February 1, 2008

Abstract

We discuss the stability of vacua in two-dimensional gauge theory for any simple, simply connected gauge group. Making use of the representation of a vacuum in terms of a Wilson line at infinity, we determine which vacua are stable against pair production of heavy matter in the adjoint of the gauge group. By calculating correlators of Wilson loops, we reduce the problem to a problem in representation theory of Lie groups, that we solve in full generality.

1 Introduction

Two-dimensional gauge theory has often been used as a toy model for four-dimensional gauge theory. Although there are no propagating

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degrees of freedom in two dimensions, two-dimensional gauge theory (supplemented with appropriate matter) can mimick some of the most intriguing features of QCD in 4 dimensions, like confinement, a fermion condensate, a non-trivial vacuum structure and stringy behavior at large N (see e.g. [1]).

In this letter we study two-dimensional gauge theory for a simple, simply connected gauge group G, with massive matter transforming in the adjoint. The effective gauge group is then G/Z, where Z is the center of the group G. The non-trivial vacuum structure can be gathered from homotopical arguments. Just as $\Pi_3(SU(3)) = \mathbf{Z}$ leads to θ-vacua in four dimensions, so $\Pi_1(G/Z) = Z$ makes $\#Z$ θ-vacua in two dimensions plausible [2]. In this letter, we want to study the existence and stablity of these vacua in more detail – the dynamics involved in the stability of the vacua cannot be read from the topological reasoning.

There are no propagating degrees of freedom for a gauge field in two dimensions. Therefore, to study dynamical aspects of the vacua, we introduce massive adjoint matter (that can mimick aspects of the dynamics of transverse gluons in higher dimensions (see e.g. [3])). We will study the stability of vacua against pair creation, by representing them as Wilson lines at infinity [4]. We have a discrete choice for the representation at infinity, since the gauge group is a simple, simply connected Lie group. In the limit where the adjoint particles are heavy, we can evaluate their interaction by computing the correlation function of the corresponding Wilson line with the Wilson line at infinity. We will study whether the strings stretching between our heavy probes have positive or negative tension. If a negative string tension is generated, this implies that the choice for representation for the vacuum at infinity is unstable against pair creation [4] [5]. Moreover, as we will see, we also have to check the stability of the vacuum against the creation of more than one particle anti-particle pair to find agreement with topological arguments [6].

These phycical problems reduce to neat problems in representation theory of Lie groups. We solve these problems in all generality, completing the analysis performed in [6].

2 Vacua as Wilson lines at infinity

In two-dimensional gauge theory, we can represent a choice of vacuum by a choice for the representation for a Wilson line at infinity [4]. This is perhaps most familiar in two-dimensional abelian gauge theory, where charges at infinity mimick the existence of θ-vacua that are associated to the non-trivial homotopy group $\Pi_1(U(1)) = \mathbf{Z}$. This is easily demonstrated since, for a $U(1)$ gauge theory with Wilson loop of charge $\frac{\theta}{2\pi}$ at infinity, we have:

$$\begin{aligned} < W_\infty(\theta) > &= \int dA\, e^{\int F \wedge * F} e^{\frac{\theta}{2\pi} \oint_\infty A} \\ &= \int dA\, e^{\int F \wedge * F + \frac{\theta}{2\pi} \int F}. \end{aligned} \tag{1}$$

Therefore, correlation functions evaluated with the insertion of a Wilson loop at infinity, or in a θ-vacuum, are identical.

For our non-abelian gauge theory with simple gauge group G, there is a similar representation, but the choice of Wilson loop is the discrete choice of irreducible representation in which we evaluate the trace of the path-ordered exponential [4].

3 Adjoint matter and stability of vacua

We introduce adjoint matter into the system, to obtain non-trivial dynamics that may be taken to mimick the behavior of the transverse gluons in higher dimensional Yang-Mills theories. We will take these particles to be heavy, such that we can measure their interaction by evaluating the expectation value of the associated Wilson loop, in the adjoint of the gauge group [7].

The Wilson loop calculation is straightforward, using standard techniques in two-dimensional gauge theory [8] [9] [10] [11] [6]. To calculate the correlator of the two Wilson loops, we glue a Wilson line at infinity in an arbitrary representation $R(\mu)$ with highest weight μ, into the propagator on the cylinder, and at the end of the cylinder, we put another Wilson loop in the adjoint $R(\theta)$, representing the particle anti-particle pair, and then we close the two-dimensional plane using the propagator on the disc. [1] We denote by g the Yang-Mills coupling,

[1] These operations are systematically explained in [11]. We heavily make use of the invariance of 2d Yang-Mills theory under area preserving diffeomorphisms.

by A_2 the area enclosed by the inner Wilson loop in the adjoint and by A_1 the area of the rest of the plane. $C(\lambda)$ is the quadratic Casimir evaluated in the representation with highest weight λ. We have then: [2]

$$
\begin{aligned}
< W_\infty(\mu) W(\theta) > &= \sum_{\lambda,\lambda'} \int_G dg\, dg' \chi_\mu(g)\chi_\lambda(g^{-1}) e^{-\frac{g^2}{2}C(\lambda)A_1} \\
& \quad \chi_\lambda(g')\chi_\theta(g')\chi_{\lambda'}((g')^{-1}) e^{-\frac{g^2}{2}C(\lambda')A_2} d(\lambda') \\
&= \sum_{\lambda'} \int_G dg' e^{-\frac{g^2}{2}C(\mu)A_1} \chi_\mu(g')\chi_\theta(g') \\
& \quad \chi_{\lambda'}((g')^{-1}) e^{-\frac{g^2}{2}C(\lambda')A_2} d(\lambda') \\
&= \sum_{\nu_i \subset \mu \otimes \theta} e^{-\frac{g^2}{2}C(\mu)A_1} e^{-\frac{g^2}{2}C(\nu_i)A_2} d(\nu_i) \\
&= \sum_{\nu_i \subset \mu \otimes \theta} d(\nu_i) e^{-\frac{g^2}{2}C(\mu)(A_1+A_2)} e^{-\frac{g^2}{2}(C(\nu_i)-C(\mu))A_2}.
\end{aligned}
$$

After we factored out the contribution corresponding to the exterior Wilson loop at infinity, we can interpret the exponent $C(\nu_i) - C(\mu)$ as the string tension for the string stretching between the adjoint matter.[3] We need $C(\nu_i) \geq C(\mu)$ for all $\nu_i \subset \mu \otimes \theta$ [4] to have a vacuum that is stable against pair creation. In the next section, we enumerate the representations $R(\mu)$ satisfying this condition, and we prove that our list is complete.

But this is not the end of the story. In fact, we also want to consider whether the vacuum is stable against the creation of more than one particle pair [6]. After a calculation similar to the one we performed in detail above, this boils down to the following extra condition: $C(\nu_i) \geq C(\mu)$ for all $\nu_i \subset \mu \otimes (\theta)^n$, where n is the (arbitrary) number of matter pairs that are created in the process of screening the trial vacuum at infinity.

Following [6] we call the vacua that are stable against the appearance of a single pair of adjoint particles metastable. The vacua

[2] We abuse notation by indicating a representation by its highest weight, for brevity. For notations, see also appendix A.

[3] This intuitive picture was corroborated in [5] by the corresponding detailed Hamiltonian analysis.

[4] The notation $\subset$ indicates that the irreducible representation ν_i is part of the decomposition of the tensor product in irreps.

satisfying the more stringent condition are the stable vacua. In the next section, by solving these problems in representation theory, we give a complete list of metastable and stable vacua, and prove the list [6] in all generality.

4 Stable and metastable vacua

4.1 Metastable vacua

Problem: Find all representations μ such that

$$C(\nu_i) \geq C(\mu) \quad \text{for all} \quad \nu_i \subset \mu \otimes \theta. \tag{2}$$

We call these representations and corresponding highest weights μ metastable.

Answer. The metastable weights for gauge group G are given by the level 1 weights of the corresponding affine (non-twisted) Kac-Moody group $\hat{G}$. [5] In appendix A and B we give our conventions and in appendix B we list the level 1 weights. To prove this statement, we first introduce the notion of a 'bad root'. We call a positive root α bad for a representation μ (with labels $m_i = \mu.\check{\alpha}_i$) if $\alpha - (m_i+1)\alpha_i$ is not a root or 0 for all i, and $(\mu+\rho).\check{\alpha} > 1$. Here and further $\alpha_1, \ldots, \alpha_r$ are the simple roots and $\check{\alpha} = \frac{2\alpha}{\alpha^2}$

We claim that μ is metastable if and only if there are no positive bad roots for μ. Indeed, all highest weights ν_i in the decomposition are of the form $\mu - \alpha$, where α is a root or 0. The inequality in (2) becomes then:

$$\begin{aligned} |\nu_i + \rho|^2 &\geq |\mu + \rho|^2 \Leftrightarrow \\ (\rho + \mu).\check{\alpha} &\leq 1. \end{aligned} \tag{3}$$

If $\alpha \leq 0$, this inequality always holds. If $\alpha > 0$, then by the theorem in appendix A, $\mu - \alpha = \nu_j$ for some j if and only if $\alpha - (m_i+1)\alpha_i$ is not a root or 0 (where m_i are the labels of μ). In that case the inequality (3) must hold. This proves our claim. We are now prepared to prove the answer in two steps:

Step 1: All weights satisfying (2) are fundamental or 0.

a) No weight μ has Dynkin label 2 or more. Indeed, it is clear that α_i would be a bad root for μ.

[5] For nomenclature, see also [12] [13].

b) No two Dynkin labels equal to 1 are allowed. Suppose $m_i = 1 = m_j \quad (i \neq j)$. Then the root $\alpha = \alpha_i + \alpha_{i+1} \ldots + \alpha_{j-1} + \alpha_j$, that connects dot i with dot j in the Dynkin diagram of G, is a bad root. [6]
We conclude that only fundamental weights can satisfy condition (2).
Step 2: Firstly, observe that $\mu = \theta$, the highest root, is not a metastable weight (since $0 \subset \theta \otimes \theta$). Secondly, if μ is not metastable for a subdiagram of the Dynkin diagram of G, then μ is not metastable for G (due to the above claim). These two observations along with Table 1 and Figure 1 show that all non level 1 fundamental weights λ_i are not metastable, except for λ_2 for F_4. The latter is not metastable since the root $\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ is bad for λ_2. That the level 1 weights do satisfy the criterium (2), can be checked on a case by case basis by going over all positive roots (listed e.g. in [14]) and verifying that all of them are good. (For miniscule weights, in particular in all simply-laced cases, this also follows from the geometric argument given in section 4.2.)

4.2 Stable vacua

Problem: Find all representations μ such that

$$C(\nu_i) \geq C(\mu) \quad \text{for all} \quad \nu_i \subset \mu \otimes \theta^n \quad \text{for any} \quad n \tag{4}$$

Answer. The stable vacua are given by the miniscule weights of G.[7]
We prove this in two steps.
Step 1: We rule out the level 1 non-miniscule weights. We distinguish the following chains of representations:
For G_2: [8]

$$\begin{aligned} \lambda_1 \otimes \lambda_2 &\supset 2\lambda_1 \\ 2\lambda_1 \otimes \lambda_2 &\supset \lambda_2 \\ \lambda_2 \otimes \lambda_2 &\supset 0 \end{aligned} \tag{5}$$

For F_4:

$$\lambda_1 \otimes \lambda_4 \supset \lambda_2$$

[6]The indices do not necessarily differ by one. See appendix B for our conventions for labeling simple roots.

[7]For nomenclature see [15] [16] and appendix B. There are $\#Z$ miniscule weights.

[8]These decompositions were obtained using the software [17]; they can also be obtained using the theorem in appendix A.

$$
\begin{aligned}
\lambda_2 \otimes \lambda_4 &\supset 2\lambda_1 \\
2\lambda_1 \otimes \lambda_4 &\supset \lambda_4 \\
\lambda_4 \otimes \lambda_4 &\supset 0
\end{aligned} \tag{6}
$$

For B_r: From the tables in [14], we find that, starting with λ_1, tensoring with the adjoint n times contains λ_{2n+1}, untill we hit the end of the Dynkin diagram. There, regardless of whether r is odd or even, we find that the representation $2\lambda_r$ is contained in the decomposition of $\lambda_1 \otimes \theta^{[r/2]}$. Tensoring $2\lambda_r$ with the adjoint brings us back over the complementary fundamental weights to λ_2. Tensoring once more with the adjoint, which is λ_2, give us a trivial representation.
For C_r: Using the tables in [14] or the theorem in appendix A, we find that generically $\lambda_k \otimes 2\lambda_1 \supset \lambda_1 + \lambda_{k+1}$. Similarly $(\lambda_1 + \lambda_{k+1}) \otimes 2\lambda_1 \supset \lambda_{k+2}$. Therefore all fundamental weights with even index are linked by chains of representations, as are all fundamental weights with odd index. Note also that $\lambda_2 \otimes 2\lambda_1 \supset 2\lambda_1$. It is clear then that all odd fundamental weights are screened to the miniscule weight λ_1, and all even to the trivial weight 0, since these have smaller Casimir then all other fundamental weights. (This will also be clear from the geometrical proof in step 2).
Step 2: We proof geometrically that all miniscule weights satisfy condition (4). The fundamental alcove A is the part of the fundamental chamber C that satisfies $\mu.\check{\alpha}_i \geq 0$ and $\mu.\check{\theta}' \leq 1$, where θ' is the highest short root. It intersects with the dominant weights only by the miniscule weights [15]. Clearly it is sufficient to prove that if $\mu + \alpha$ lies in the fundamental chamber C, for μ miniscule and α a nonzero in the root lattice Q, then:

$$
\begin{aligned}
|\mu + \alpha + \rho|^2 &> |\mu + \rho|^2 \quad \text{or} \\
(2\mu + 2\rho + \alpha).\alpha &> 0.
\end{aligned} \tag{7}
$$

Now, C is covered by alcoves $\nu + A$, where $\nu \in Q \cap C$. Since $\mu \in A$ and $\mu + \alpha \in C$, we conclude that $\alpha \in C$. Since the inverse of the Cartan matrix has only positive entries, we conclude that α is a linear combination of simple roots with non-negative coefficients. Hence $\mu.\alpha \geq 0$, $(\mu + \alpha).\alpha \geq 0$, and $\rho.\alpha > 0$. QED.

5 Conclusion

We proved that the classification of stable vacua in two-dimensional gauge theory with simple, simply connected gauge group G and heavy matter in the adjoint, in an algebraic fashion based on Wilson loop computations, agrees with the naive homotopical arguments for the classification of vacua. The stable vacua correspond to the miniscule weights of G, of which there are $\#Z$, where Z is the center of G. Moreover we gave a proof of a complete list of vacua that are metastable, i.e. stable against the creation of a single adjoint particle anti-particle pair. These are the fundamental weights that are level 1 weights for affine (non-twisted) Kac-Moody algebras. This completes the partial proof in [6].

From the set-up it is clear that we might as well have introduced other matter in a representation S, and that the same analysis would have boiled down to the following mathematical questions: for which representations R is $C(R) \leq C(R_i)$ for all $R_i \subset R \otimes S$, and similarly, for which representation R is $C(R) \leq C(R_i)$ for all $R_i \subset R \otimes S^n$? This question seems to be of less physical importance, since matter in a general representation S cannot be expected to mimick aspects of transverse gluons in higher dimensions, but the question seems mathematically interesting, in view of the fact that it has such a non-trivial answer for the adjoint representation. The techniques in this paper should carry over, certainly, to the case of representations R where the multiplicities of all non-zero weights is 1.

Physically more interesting is the question of stability of vacua in the presence of adjoint matter and extra particles in a fundamental representation of the gauge group. It should be easy to give the answer to this question algebraically, starting from the results obtained in this letter. (There will be a similar parallel topological argument, based on the representation of the center of the gauge group in the chosen fundamental representation.)

It would be very nice to find a physical interpretation for the intriguing fact that all metastable vacua are exactly the level 1 weights of affine (non-twisted) Kac-Moody algebras (see also the suggestion in [6] of a phase transition at finite mass). Moreover, it seems interesting to generalize the explicit study of the different vacua, and instanton contributions in 2d QCD (see e.g. [18]) for different gauge groups. In fact, there seems to be a broad class of questions in two-dimensional gauge theory which have only been studied in detail for

$SU(N)$ gauge groups that should have interesting counterparts for more general groups.

Acknowledgements

This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FC02-94ER40818.

A A theorem

We denote by W the Weyl group, by Q the root lattice, by C the fundamental chamber, and by ρ half the sum of all positive roots. By A we denote the fundamental alcove and λ_i are the fundamental weights, corresponding to the simple roots α_i. e_{α_i} is the Lie algebra generator associated with the simple root α_i.

Theorem (see e.g. [14]): The multiplicity of a representation $R(\nu)$ with highest weight $\nu = \lambda + \mu'$ in the tensor product of the representation $R(\lambda)$ with highest weight λ and Dynkin labels $m_i = \lambda.\check{\alpha}_i$, and $R(\mu)$, where μ' is a weight of $R(\mu)$, is equal to the dimension of the subspace of the weightspace of $R(\mu)$ with weight μ' consisting of vectors killed by $(e_{\alpha_i})^{m_i+1}$ for all simple roots α_i.

B Miniscule and level 1 weights

The miniscule weights are the non-zero dominant weights in the fundamental alcove. They can be defined by $\mu.\check{\alpha}_i \geq 0$ and $\mu.(\check{\theta'}) \leq 1$. If we take, instead of $W \times Q$, the group $W \times Q'$ where Q' is the $\mathbf{Z}$-span of the long roots, then the fundamental alcove becomes bigger: the inequality $\mu.\check{\theta}' \leq 1$ is replaced by $\mu.\check{\theta} \leq 1$, and intersects the dominant weights by all level 1 weights of the affine (non-twisted) Kac-Moody algebra.

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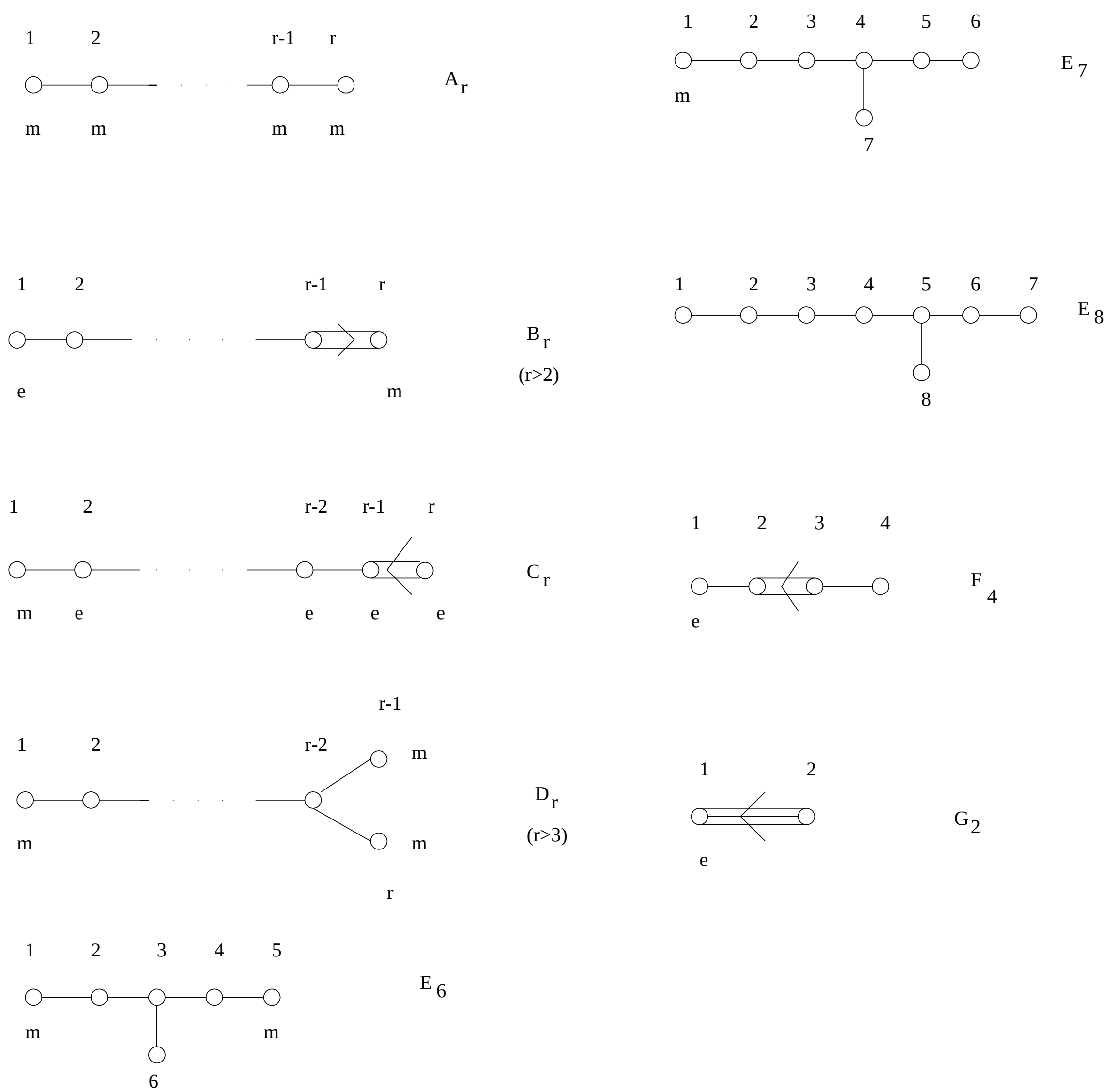

Figure 1: *We depict the Dynkin diagrams for all simple Lie algebras along with a numbering of nodes, the non-zero miniscule weights (m), which are all level 1, as well as the extra level 1 weights (e). All of them are fundamental weights and we indicate the corresponding node in the Dynkin diagram.*

	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
# level 1	r+1	3	r+1	4	3	2	1	2	2
# miniscule	r+1	2	2	4	3	2	1	1	1
θ	$\lambda_1 + \lambda_r$	λ_2	$2\lambda_1$	λ_2	λ_6	λ_6	λ_1	λ_4	λ_2

Table 1: *We table the number of level 1 and miniscule weights for the finite simple compact Lie algebras, and express their highest root θ in terms of the fundamental weights.*

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